



Fixed point theorems for mappings with convex diminishing diameters on cone metric spaces

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ABSTRACT

In this work, Cantor's intersection theorem is extended to cone metric spaces and as an application, a fixed point theorem is derived for mappings with locally power diminishing diameters.

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1. Introduction and preliminaries

Cone metric spaces were rediscovered by Huang and Zhang [1] who replaced the set of real numbers by an ordered Banach space in the definition of the metric, and obtained some fixed point theorems for contractive type mappings. Although this notion was introduced in the middle of the 20th century, Huang and Zhang [1] defined the convergence via interior points of the cone P , by which the order in Banach space E is defined. This approach allows the investigation of cone spaces also in the case where the cone is not necessarily normal, which was not possible before 2007. Since then, there have appeared many papers containing interesting fixed point results in cone metric spaces. In this work, we extend Cantor's intersection theorem to cone metric spaces and derive, as an application, a fixed point theorem for mappings with locally power diminishing diameters, which were introduced by Istrăţescu [2].

Let E be a real Banach space and P a subset of E . P is called a cone if and only if:

- (i) P is closed and non-empty, and $P \neq \{\theta\}$;
- (ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;
- (iii) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering \leq on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ (the interior of P).

A cone $P \subset E$ is called normal if there is a number $K \geq 1$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

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The least positive number satisfying the above inequality is called the normal constant of P . A cone P is regular if every increasing sequence which is bounded from above is convergent, i.e., if $\{x_n\}$ is a sequence such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y$$

for some $y \in E$, then there is an element $x \in E$ such that $\|x_n - x\| \rightarrow 0 (n \rightarrow \infty)$. Equivalently, the cone P is regular if and only if a decreasing sequence which is bounded from below is convergent. P is called a minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in E$, and strongly minihedral if every subset of E which is bounded from above has a supremum [3]. Every strongly minihedral cone is normal [3].

In the following we always suppose that E is a Banach space, P is a solid cone, i.e., $(\text{int}P \neq \emptyset)$ in E and \preceq is a partial ordering with respect to P .

Definition 1 ([1]). Let X be a non-empty set. Suppose that $d : X \times X \rightarrow E$ satisfies:

- (d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space.

Example 1 ([1]). Suppose that $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, and $X = \mathbb{R}$, and let $d : X \times X \rightarrow E$ be such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is constant. Then (X, d) is a cone metric space.

Definition 2 ([1]). Let (X, d) be a cone metric space, suppose that $x \in X$, and let $\{x_n\}_{n \geq 1}$ be a sequence in X . Then:

- (i) $\{x_n\}_{n \geq 1}$ converges to x whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$, for all $n \geq N$. We denote this by $x_n \rightarrow x (n \rightarrow \infty)$ or $\lim_{n \rightarrow \infty} x_n = x$.
- (ii) $\{x_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$, for all $n, m \geq N$.
- (iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Theorem 1 ([1]). Let (X, d) be a complete normal cone metric space with a normal and solid cone P . Suppose that the mapping $T : X \rightarrow X$ is a contraction. Then T has a unique fixed point in X , and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Remark 1 ([4] Proposition (2.2)).

- (1) $c \in \text{int}P \Leftrightarrow [-c, c]$ is a neighborhood of θ in norm topology in Banach space E ;
- (2) $[-c, c] = (c - P) \cap (P - c)$; $\text{int}P = (c - \text{int}P) \cap (\text{int}P - c)$.

Remark 2 ([5]).

- (1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
- (2) If $\theta \preceq u$ and $u \ll c$ for each $c \in \text{int}P$ then $u = \theta$.

Remark 3 ([6]). If E is a real Banach space with cone P and if $a \preceq \lambda a$ where $a \geq \theta$ and $0 \leq \lambda < 1$, then $a = \theta$.

Definition 3 ([7]). Let (X, d) be a cone metric space. Then $A \subset X$ is called bounded above if there exists $c \in E$, $c \gg \theta$, such that $d(x, y) \preceq c$, for all $x, y \in A$, and is called bounded if $\delta(A) = \sup\{d(x, y) : x, y \in A\}$ exists in E . If the supremum does not exist, we say that A is unbounded.

For more details on the subject, we refer the reader to [4–13].

2. Results

Let X be a complete cone metric space with cone metric d over a solid cone P .

Definition 4. A mapping $T : X \rightarrow X$ is said to have locally convex diminishing diameters of infinite order if there exists a sequence of positive numbers $(a_i)_0^\infty$ with $\sum a_i < 1$, and for each bounded set A in X there exists an integer $n = n(A)$ such that the following inequality holds:

$$\delta(T^n(A)) \preceq a_0 \delta(A) + a_1 \delta(T(A)) + a_2 \delta(T^2(A)) + \cdots + a_{n-1} \delta(T^{n-1}(A)).$$

If $a_{i+1} = 0$, for all $i = 0, 1, 2, \dots$, then T has locally power diminishing diameters.

Proposition 1. Let $T : X \rightarrow X$ be a mapping satisfying the property that there exists $k \in [0, 1)$ and, for each $x \in X$, there exists an integer $n = n(x)$ with the property that for all $y \in X$,

$$d(T^n(x), T^n(y)) \leq kd(x, y).$$

Then T has locally power diminishing diameters.

Proof. Consider an integer m such that $2k^m < 1$. Let A be a bounded set in X for which there exists $c \in \text{int}P$ such that $d(x, y) \leq c$ for all $x, y \in A$. Now, suppose that $x \in A$ and consider the following sequence of integers: $n_1 = n(x)$, $x_1 = T^{n_1}(x)$, $n_2 = n(x_1)$, $x_2 = T^{n_2}(x_1)$, \dots , $n_{m-1} = n(x_{m-1})$, $x_{m-1} = T^{n_{m-1}}(x_{m-1})$. Now if x, y are arbitrary points in A we have

$$\begin{aligned} d(T^{n_1+\dots+n_m}(x), T^{n_1+\dots+n_m}(y)) &\leq d(T^{n_1+\dots+n_m}(x), T^{n_1+\dots+n_m}(z)) + d(T^{n_1+\dots+n_m}(z), T^{n_1+\dots+n_m}(y)) \\ &\leq kd(T^{n_1+\dots+n_{m-1}}(x), T^{n_1+\dots+n_{m-1}}(z)) + kd(T^{n_1+\dots+n_{m-1}}(z), T^{n_1+\dots+n_{m-1}}(y)) \\ &\leq k^2 d(T^{n_1+\dots+n_{m-2}}(x), T^{n_1+\dots+n_{m-2}}(z)) + k^2 d(T^{n_1+\dots+n_{m-2}}(z), T^{n_1+\dots+n_{m-2}}(y)) \\ &\vdots \\ &\leq k^{m-1} d(T^{n_1}(x), T^{n_1}(z)) + k^{m-1} d(T^{n_1}(z), T^{n_1}(y)) \\ &\leq k^m d(x, z) + k^m d(z, y) \\ &\leq 2k^m \delta(A). \quad \square \end{aligned}$$

Definition 5 (Property (C)). A sequence $\{a_n\}$ in ordered Banach space E has property (C) if for each $c \in \text{int}P$ there exists $k \in \mathbb{N}$ such that for all $n > k = k(c)$ we have that $a_n \ll c$. If $a_n \rightarrow \theta$ in Banach space E then $\{a_n\}$ has property (C).

Note that a sequence $\{a_n\}$ in a normal cone P of a Banach space E has property (C) if and only if $\lim_{n \rightarrow \infty} a_n = \theta$ in $(E, \|\cdot\|)$.

Necessity: Assume $\{a_n\}$ has property (C) and let $\epsilon > 0$ be given. Then there exists $c \gg \theta$ such that $K\|c\| \leq \epsilon$, where K is the normal constant. By property (C), we can find n_0 such that $a_n \ll c$ for all $n > n_0$. Then by the normality of P , $\|a_n\| \leq K\|c\| \leq \epsilon$. That is, $\lim_{n \rightarrow \infty} a_n = \theta$.

Conversely, assume that $\lim_{n \rightarrow \infty} a_n = \theta$ and let $c \gg \theta$ be given. We can find $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$. By the assumption, there exists n_0 such that $\|a_n\| < \delta$ for all $n > n_0$. Hence, $a_n \ll c$ for all $n > n_0$.

The following is an alternative proof of Lemma 6 in [7].

Lemma 1. Let (X, d) be a cone metric space, P a strongly minihedral cone of X with $\text{int}P \neq \emptyset$ and A a bounded subset of X . Then \bar{A} is bounded and $\delta(A) = \delta(\bar{A})$.

Proof. Suppose that $a, b \in \bar{A}$ and let $\theta \ll c$ be given. Then there exist sequences $a_n, b_n \in A$ such that $a_n \xrightarrow{d} a$, $b_n \xrightarrow{d} b$ and then there exists $k \in \mathbb{N}$ such that

$$d(a_n, a) + d(b_n, b) \ll c \quad (2.1)$$

for each $n > k = k(c)$. In this case we have

$$\begin{aligned} d(a, b) &\leq d(a, a_n) + d(a_n, b_n) + d(b_n, b) \\ &= d(a_n, a) + d(b_n, b) + d(a_n, b_n) \\ &\leq d(a_n, a) + d(b_n, b) + \delta(A). \end{aligned} \quad (2.2)$$

From (2.1) and (2.2), it follows that

$$d(a, b) \ll c + \delta(A).$$

Because $a \ll b + c$ for each $c \in \text{int}P$ it follows that $a \leq b$; so we have that for each $a, b \in \bar{A}$, $d(a, b) \leq \delta(A)$. Hence, the subset $\{d(x, y) : x, y \in \bar{A}\}$ is bounded above with $\delta(A) \in P$. Since the cone P is strongly minihedral, then \bar{A} is bounded and $\delta(\bar{A}) \leq \delta(A)$. Further, the proof that $\delta(A) \leq \delta(\bar{A})$ is obvious. \square

The following lemma is due to Abdeljawad et al. ([14], Lemma 10).

Lemma 2. Every Cauchy sequence of a cone metric space (X, d) over a strongly minihedral cone is bounded.

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, d) . Then for some $\theta \ll c$ there exists n_0 such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$. Since P is strongly minihedral then there exists $c_1 = \sup\{c, d(x_n, x_m) : m, n \leq n_0\}$, that is, $d(x_n, x_m) \leq c_1$ for all m, n . Since P is strongly minihedral, we have that

$$\sup\{d(x_n, x_m) : m, n \in \mathbb{N}\}$$

exists. As a result, $\{x_n\}$ is bounded. \square

The following is Cantor's intersection theorem for cone metric spaces.

Theorem 2. A cone metric space (X, d) over a strongly minihedral solid cone P is complete if and only if every decreasing sequence $\{A_n\}$ of a non-empty closed subset of X for which the sequence of diameters $\delta(A_n) \rightarrow 0$ as $n \rightarrow \infty$ has a non-empty intersection.

Proof. For each $n \in \mathbb{N}$, suppose that $x_n \in A_n$. Let $\theta \ll c$ be given. According to Definition 5 and the remark after it, the sequence of diameters $\{\delta(A_n)\}$ has property (C), so we can find $N_1 \in \mathbb{N}$ such that $\delta(A_n) \ll c$ for all $n > N_1$. Since A_n is decreasing then

$$d(x_n, x_m) \leq \delta(A_n) \quad \text{for all } m \geq n \text{ and } n \in \mathbb{N}. \quad (2.3)$$

Hence, by Remark 2(1), we have that $d(x_n, x_m) \ll c$ for all $m \geq n > N_1$. Therefore, $\{x_n\}$ is Cauchy. Since (X, d) is complete, there exists $x \in X$ such that $x_n \rightarrow x$. To show that $x \in \bigcap_{n=1}^{\infty} A_n$, suppose that $m \rightarrow \infty$ in (2.3) and by normality of the cone and the assumption that each A_n is closed, we conclude that $d(x_n, x) \leq \delta(A_n)$. This means that $x \in A_n$ for each n .

Conversely, let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in X and $\theta \ll c$ be given. Define $A_n = \{x_k : k \geq n\}$. Then clearly $\overline{A_{n+1}} \subset \overline{A_n}$. Also since $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, then there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll \frac{c}{2}$ for all $n, m \geq N$. Now, we have that $\delta(\overline{A_n}) = \delta(A_n) \leq \frac{c}{2} \ll c$, i.e., $\delta(\overline{A_n}) = \delta(A_n) \ll c$ for all $n \geq N$, $\delta(A_n) \ll c$. Hence $\delta(\overline{A_n}) \rightarrow \theta$ as $n \rightarrow \infty$. Since $\bigcap_{n=1}^{\infty} \overline{A_n} \neq \emptyset$, there is an $x \in \overline{A_n}$ for each n . Now we have

$$d(x_n, x) \leq \delta(\overline{A_n}) \rightarrow \theta \quad \text{as } n \rightarrow \infty,$$

i.e., $x_n \rightarrow x$ as $n \rightarrow \infty$. Therefore (X, d) is a complete cone metric space. \square

Example 2. Suppose that $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, and $X = \mathbb{R}$ and let $d : X \times X \rightarrow E$ be such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is constant. Then (X, d) is a cone metric space. Suppose that $A_n = [0, \frac{1}{n}] \subset X$. The cone P is strongly minihedral. Then, we have

$$\delta(A_n) = \sup\{d(x, y) : x, y \in A_n\} \leq \sup\left\{\left(\frac{2}{n}, \frac{2\alpha}{n}\right), n \in \mathbb{N}\right\} = \frac{2}{n}(1, \alpha) \rightarrow \theta, \quad \text{as } n \rightarrow \infty.$$

Since $A_{n+1} \subset A_n$, according to Theorem 2, we have that $\bigcap_{n=1}^{\infty} A_n$ has a non-empty intersection.

Remark 4. In the previous theorem, $\bigcap_{n=1}^{\infty} A_n = \{x\}$, i.e., the intersection is a singleton. Indeed, suppose that $z \neq x$ and $z \in A_n$ for each n . Let $\theta \ll c$ be given. Then $d(x, z) \leq \delta(A_n)$ for all $n \in \mathbb{N}$ and $\delta(A_n) \ll c$ for all $n > k = k(c)$. Hence $d(x, z) \ll c$ (by Remark 2(2)) and $d(x, z) = \theta$, i.e., $x = z$, which is a contradiction.

Theorem 3. Let (X, d) be a complete cone metric space over a strongly minihedral and solid cone P and $T : X \rightarrow X$ a continuous mapping with locally power diminishing diameters. If X is bounded, then T has a unique fixed point.

Proof. Since T is a mapping with locally power diminishing diameters, there exists $n_1 = n(X)$, $n_2 = n(X_1)$, \dots , $n_m = n(X_{m-1})$ such that

$$X_1 = \overline{T^{n_1}(X)}, \quad X_2 = \overline{T^{n_2}(X_1)}, \dots, X_m = \overline{T^{n_m}(X_{m-1})}.$$

So we have

$$\begin{aligned} \delta(X_1) &= \delta(\overline{T^{n_1}(X)}) \leq k\delta(X). \\ \delta(X_2) &= \delta(\overline{T^{n_2}(X_1)}) \leq k\delta(\overline{T^{n_1}(X)}) \leq k^2\delta(X). \\ \delta(X_3) &= \delta(\overline{T^{n_3}(X_2)}) \leq k\delta(\overline{T^{n_2}(X_1)}) \leq k^2\delta(\overline{T^{n_1}(X)}) \leq k^3\delta(X). \end{aligned}$$

Similarly we get

$$\delta(X_m) = \delta(\overline{T^{n_m}(X_{m-1})}) \leq k^m\delta(X) \rightarrow \theta \quad \text{as } m \rightarrow \infty.$$

Hence, for given $c \in \text{int}P$ there exists $N = N(c) \in \mathbb{N}$ such that $k^m\delta(X) \ll c$ for all $m \geq N$. According to Remark 2 (1) we have that $\delta(X_m) \ll c$ for all $m \geq N$. Since the diameter of any bounded set A satisfies $\delta(A) = \delta(\overline{A})$ (by Lemma 2) we have obtained that the sequence of diameters $\delta(X_m) = \delta(\overline{T^{n_m}(X_{m-1})})$ has property (C) (because each strongly minihedral cone is normal (see [3]); this is equivalent to $\delta(X_m) \rightarrow \theta$ as $m \rightarrow \infty$). Thus we obtain the sequence $\{\overline{T^{n_m}(X_{m-1})}\}$ of non-empty, decreasing, closed and bounded subsets of X . Since (X, d) is complete then by Cantor's intersection theorem on cone metric spaces (Theorem 2), $\bigcap_{m=1}^{+\infty} \overline{T^{n_m}(X_{m-1})} = \{z\}$; this implies that for any $x \in X$, the iterative sequence $\{T^{n_m}x\}$, $m = 1, 2, \dots$, converges to z . Since T is continuous, $z = \lim T^{n_m}x = \lim T^{n_m+1}x = T(\lim T^{n_m}x) = Tz$, i.e., z is a fixed point of T . According to Remark 4, uniqueness of the fixed point follows. \square

Corollary 1. Let (X, d) be a complete cone metric space and $T : X \rightarrow X$ a continuous mapping which satisfies the property that there exists $k \in [0, 1)$ and for each $x \in X$ there exists an integer $n = n(x)$ with the property that for all $y \in X$,

$$d(T^n(x), T^n(y)) \leq kd(x, y).$$

If X is bounded, then T has a unique fixed point.

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